

# A NATURAL BIAS APPROACH TO CARDINAL SPLINE CURVES\*

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The cardinal spline approach to defining interpolatory curves has been of recent interest in the application of computer graphics to modelling and animation problems [1], [2]. In our formulation of this interpolation problem (C) we desire to obtain a vector valued function  $V$  which forms a  $C^1$  map from  $[0,n]$  into the plane in such a way that  $V(i)$  and  $V'(i)$  attain prespecified (vector) values  $P_i$  and  $T_i$  for the integers  $i=0, 1, \dots, n$ . The intent is that the planar curves  $V(s)$  pass through each point  $P_i$  in the direction corresponding to  $T_i$ . In actual practice it is common for only the interpolation points  $P_i$  to be user specified. Suitable tangents  $T_i$  are then provided by some heuristic to complete the specification of (C). If, for instance, the  $T_i$  values are computed as

$$T_i = (P_{i+1} - P_{i-1})/2 \quad (1)$$

(each tangent is the average of the incoming and outgoing chord vectors to the adjacent points), then the solution to (C) is the Catmull-Rom spline. Once the tangents  $T_i$  have been defined we may write the cardinal spline curve  $V(s)$  as a collection of segments

$$V(s) = V_i(s-i) \quad i \leq s \leq i+1, i=1, \dots, n-1$$

where each curve segment  $V_i$  is a Hermite interpolant which may be written simply in terms of the cardinal Hermite basis. Defining a vector function of a real variable  $s$  by

$$H(s) = [1-3s^2+2s^3, 3s^2-2s^3, s-2s^2+s^3, -s^2+s^3]$$

we may write

$$V_i(s) = H(s) * [P_i, P_{i+1}, T_i, T_{i+1}]^t. \quad 0 \leq s \leq 1 \quad (2)$$

For most applications the magnitudes of the tangents  $T_i$  can be allowed to vary without effecting their primary function which is to furnish a suitable direction for the interpolating curve at each node. The effects of these variations on the resulting curve are perceived subjectively as "shape" changes. They do not alter the fundamental property of such curves - that they possess a continuously varying unit tangent vector. The effects are best understood by considering a single section of the curve which passes through points  $P_0$  and  $P_1$  with tangents  $T_0$  and  $T_1$ . In general, increasing the magnitude of a tangent vector forces the interpolating curve to "stay closer longer" to the line on which the tangent vector lies. Thus increasing the magnitude of one tangent at the expense of the other will "pull" the curve in its direction. This effect is called "bias." Increasing the magnitudes of both tangents proportion-

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ally, forces the bend more toward the middle; decreasing the magnitudes allows the curve to "leave the tangent directions sooner" and causes a flattening effect called "tension." A discussion of these effects is given in [3].

The important issue here is the point of view that the salient feature of the tangent vectors is the "direction" they impart to the curve and that their magnitudes may be adjusted to effect shape control. A natural question to ask in this context is the following - given the two endpoints and corresponding tangent directions which define a segment on the curve, what should be the default bias, i.e. the ratio of their magnitudes which serves as a base for further adjustments? We feel that this question has a natural answer based on the geometry of each curve segment, one that yields curves with interesting and desirable shape properties. We give a brief summary of our treatment.

Given a segment with endpoints  $P_0$  and  $P_1$  and tangents  $T_0$  and  $T_1$  we solve the equation

$$xT_0 + yT_1 = P_1 - P_0 \quad (3)$$

for scalar values  $x$  and  $y$ . There are three primary cases to consider:

- i)  $xy > 0$
- ii)  $xy < 0$
- iii) (3) has no solution

Other cases occur when  $T_0$  and  $T_1$  lie on the line containing  $P_0$  and  $P_1$  and are degeneracies which we discuss separately. For case i) we rescale  $T_0$  and  $T_1$  as  $xT_0$  and  $yT_1$  respectively. For cases ii) or iii) we solve (3) with  $T_0$  replaced by  $T_0'$ , its "reflection" in the line containing  $P_0$  and  $P_1$ . The  $x$  and  $y$  thus obtained will now satisfy  $xy > 0$ , and we proceed as before rescaling  $T_0$  (the original  $T_0$ ) as  $xT_0$  and  $T_1$  as  $yT_1$ . This rescaling is the means by which a "natural" bias for the segment is obtained. See figure 1.

When the new tangents are multiplied by a common "tension" factor  $t$ , the resulting curve segment defined by (2) has the following properties which form our justification for using the word "natural" to describe the biasing technique.

For case i)

- For  $t = 3$  the segment has zero curvature at both end points  $P_0$  and  $P_1$ . For  $t < 3$  the segment will have no inflection points (changes of concavity). For  $t > 3$  the segment will have exactly two inflection points.
- For  $t = 2$  the 3rd order terms vanish and the resulting curve is in fact a parametric quadratic.
- For  $t = 6$  the segment will have a cusp where the curve parameter  $s = .5$ .

For cases ii) and iii)

- The curve will intersect the line segment  $S$  connecting  $P_0$  and  $P_1$  exactly once (let  $Q$  denote this point).

- The maximum deviations of the curve from S on either side of Q will be the same.
- For  $t = 3$  the tangent to the curve at Q is perpendicular to S.

As a consequence of our results for segments falling under case i) we can prove the following theorem.

Let  $P_1, P_2, \dots, P_n$  be vertices of a convex planar polygon P given in order of adjacency, and let tangents  $T_i$  be defined by (1) with

$$T_0 = T_{n-1} \quad \text{and} \quad T_{n+1} = T_1$$

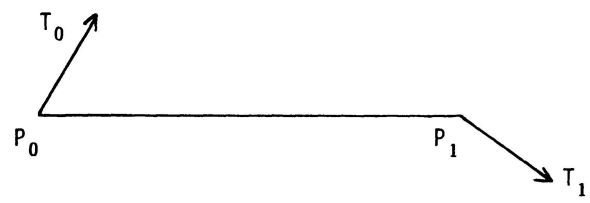
The closed cardinal spline curve (whose segments are defined by (2) where the tangents are biased by the procedure we have outlined and controlled by a tension factor  $t$ ) will form the boundary of a convex set C containing P for any value of  $t$  such that  $0 < t < 3$ .

In short, a convex control polygon will yield a convex interpolant. We can also give examples where the interpolant fails to be convex if the tangents are left unbiased.

#### References

1. Alvy Ray Smith, "Spline Tutorial Notes - Technical Memo No. 77", SIGGRAPH '83 Tutorial Notes: Introduction to Computer Animation, pp. 64-75, July, 1983.
2. D. H. Kochanek, and R. H. Bartels, "Interpolating Splines with Local Tension, Continuity, and Bias Control", SIGGRAPH '84, Minneapolis, July 23-27, 1984, pp. 33-41.
3. J. D. Foley, and A. Van Dam, Fundamentals of Interactive Computer Graphics, pp. 514-523, Addison-Wesley, 1983.

case i)



case ii)

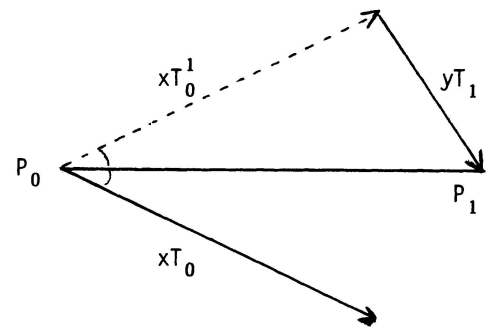
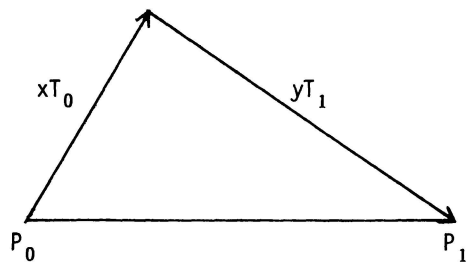
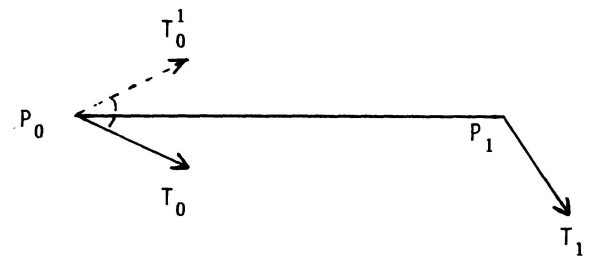


Figure 1